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ON THE DISSIPATION ASSOCIATED WITH EQUILIBRIUM SHOCKS IN FINITE--ETC(U)
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TECHNICAL REPORT NO. 38

ON THE DISSIPATION ASSOCIATED WITH EQUILIBRIUM SHOCKS IN FINITE ELASTICITY

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On the dissipation associated with equilibrium shocks in finite elasticity[1]

by

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Abstract

Equilibrium fields with discontinuous displacement gradients can occur in finite elasticity for certain materials. The presence of such "equilibrium shocks" affects the energy balance in the elastostatic field, and the present paper is concerned with a notion of dissipation associated with this energy balance. A dissipation inequality is proposed for three-dimensional equilibrium shocks for both compressible and incompressible materials. The consequences of this inequality are studied for weak shocks in plane strain for compressible materials and for shocks of arbitrary strength in anti-plane strain for a class of incompressible materials. A thermodynamic argument for the dissipation inequality is also given.

1. Introduction

An example discussed in [1] shows that it is possible for the differential equations governing finite equilibrium deformations of an elastic solid — for an otherwise "reasonable" material — to lose their ellipticity

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[1] The results communicated in this paper were obtained in the course of an investigation supported by Contract N00014-75-C-0196 between the California Institute of Technology and the Office of Naval Research.

in the presence of sufficiently severe strains. For a homogeneous, isotropic, compressible elastic solid, explicit conditions on the principal stretches and the strain energy which are necessary and sufficient for the ellipticity of the field equations of plane elastostatics were given in [2].

One of the consequences of such a loss of ellipticity is the possible occurrence of elastostatic fields in which the displacements may fail — in certain portions of the body, at least — to have continuous derivatives of even the first order. A detailed investigation of the local structure of such fields in plane finite elastostatics is described in [3], where it is pointed out that they may be relevant to the study of the "Lüders bands" commonly observed in ductile materials[1]. Indeed, deformations with abruptly changing gradients arise in a wide variety of solids in connection with what has been termed "localized shear"[2]. Boundary value problems in the equilibrium theory of finite elasticity in which locally severe deformations — near a cavity, for example — give rise to a loss of ellipticity and an associated reduction in the smoothness of the stress and displacement fields would seem to be of special interest in the study of fracture.

The mathematical description of phenomena of the kind described above has certain features in common with the theory of stationary transonic flows in inviscid gas dynamics[3]. One of the conspicuous aspects of such flows is the occurrence of shocks — surfaces across which the fluid

[1] See Chapter 18 of [4].

[2] For an extensive discussion of localized shear, see the article by Rice [5].

[3] In fact, a precise analogy exists between the boundary value problems for gas flows past an obstacle on the one hand, and finite anti-plane strain (for a certain class of incompressible elastic materials) of a medium containing a crack or cavity on the other hand. See § 7 of [6].

velocity, pressure, density and entropy suffer jump discontinuities. In the setting of finite elastostatics, the analogous surfaces are those which carry jump discontinuities in the first derivatives of the displacement vector; they will be called equilibrium shocks in the present paper.

In gas dynamics, one of the essential restrictions on the flow across a shock — whether moving or stationary — is that imposed by the requirement that the entropy of a particle shall increase as the particle crosses a shock[1]. This condition leads to major qualitative conclusions concerning the flow[2], and it also expresses — in an idealized way, to be sure — the dissipative character of the process of shock formation.

In this extensive study of weak solutions of hyperbolic systems of conservation laws, Lax[3] has emphasized the mathematical role of "entropy conditions" of various kinds in connection with the issue of singling out the physically meaningful weak solution to the initial value problem from among the many such solutions nominally admitted by the differential equations and initial conditions themselves. The many beautiful results pertaining to this question which are discribed in [8] - [11] appear to be most appropriate for dynamical problems — the propagation of shock waves, for example. They do not seem readily interpretable in the context of finite elastic equilibrium, in which time is not one of the independent variables and the initial value problem is of no apparent significance. The objective of the present paper is the derivation and discussion of a dissipation condition which is

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[1] For an extensive discussion of shocks in compressible flow, see [7].

[2] Gas dynamical shocks are always compressive, for example.

[3] See [8] - [11], where references to related work may be found.

analogous to the "increasing entropy" requirement for steady gas flows and which is thought to be appropriate for equilibrium shocks in finite elasticity. A detailed application of the result derived here may be found in [3].

The field equations of finite elastostatics for both compressible and incompressible materials are given in the following section. The class of weak solutions to be considered is described in Section 3, which also includes the jump conditions to be satisfied across an equilibrium shock. The dissipation condition for a quasi-static time-dependent family of equilibrium shocks is derived and discussed in Section 4. The implications of the dissipation condition for weak shocks in plane deformations of compressible materials are examined in Section 5. Section 6 is devoted to equilibrium shocks and the dissipation inequality in anti-plane strain for a special class of incompressible materials. A thermodynamic argument in support of the dissipation requirement is given in the final section.

2. Equations of elastostatics

Let $\mathcal{R}$ be the interior of a three-dimensional region occupied by an elastic body in its undeformed state. In a deformation of the body, $\mathcal{R}$ is mapped invertibly onto a domain $\mathcal{R}^*$, so that a particle with position vector $\underset{\sim}{x}$ in $\mathcal{R}$ is carried to the point in $\mathcal{R}^*$ whose position vector is $\underset{\sim}{y}$:

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \ , \qquad \underset{\sim}{x} \in \mathcal{R} \ . \tag{2.1}$$

For the moment, the displacement vector $\underset{\sim}{u}$ is assumed to be twice continuously differentiable on $\mathcal{R}$.

Let x_i, y_i and u_i be the components in a fixed rectangular cartesian

coordinate system of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{u}$, respectively. The tensors[1] $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{C}$ defined by

$$\mathbf{F} = \nabla \mathbf{y} = \mathbf{1} + \nabla \mathbf{u} \ , \quad \mathbf{G} = \mathbf{F}\mathbf{F}^T \ , \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \ , \qquad (2.2)^2$$

where ∇ is the gradient operator and $\mathbf{1}$ the identity tensor, have components

$$F_{ij} = y_{i,j} = \delta_{ij} + u_{i,j} \ , \quad G_{ij} = F_{ik}F_{jk} \ , \quad C_{ij} = F_{ki}F_{kj} \ ; \qquad (2.3)^3$$

$\mathbf{F}$ is the deformation gradient tensor, $\mathbf{G}$ and $\mathbf{C}$ the left- and right- deformation tensors, respectively. The Jacobian of the mapping is assumed positive:

$$J = \det \mathbf{F} > 0 \qquad \text{on } \mathcal{R} \ . \qquad (2.4)$$

If ρ and ρ_0 are the mass densities[4] in $\mathcal{R}$ and $\mathcal{R}^*$, balance of mass requires

$$\rho J = \rho_0 \qquad \text{on } \mathcal{R} \ . \qquad (2.5)$$

If σ_{ij} are the components of the nominal (or Piola) stress tensor[5] $\boldsymbol{\sigma}$, the local condition of force equilibrium in the absence of body forces

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[1] The same symbol, e.g. $\mathbf{F}$, will be used for a tensor and its matrix of components in the given coördinate system.

[2] The superscript T stands for transposition.

[3] Latin subscripts have the range 1, 2, 3 and repeated subscripts are summed over this range. A subscript preceded by a comma indicates differentiation with respect to the corresponding x- coordinate.

[4] ρ_0 is taken to be constant.

[5] The σ_{ij} represent forces per unit undeformed area.

may be expressed as

$$\sigma_{ij,j} = 0 \qquad \text{on } \mathcal{R} , \tag{2.6}$$

provided that $\underset{\sim}{\sigma}$ is continuously differentiable on $\mathcal{R}$. The components τ_{ij} of the true (or Cauchy) stress tensor[1] $\underset{\sim}{\tau}$ are related to σ_{ij} via $\underset{\sim}{F}$ as follows:

$$\tau_{ij} = \frac{1}{J}\sigma_{ik}F_{jk} . \tag{2.7}$$

If $\underset{\sim}{\tau}$ is continuously differentiable on $\mathcal{R}^*$, then (2.6) is equivalent to

$$\frac{\partial \tau_{ij}}{\partial y_j} = 0 \qquad \text{on } \mathcal{R}^* . \tag{2.8}$$

Moment equilibrium leads to $\tau_{ij} = \tau_{ji}$, but $\underset{\sim}{\sigma}$ is in general not symmetric.

For a <u>compressible</u> homogeneous elastic material with a strain energy density $W = W(\underset{\sim}{F})$ per unit undeformed volume, the relation between stress and deformation may be taken in the form

$$\sigma_{ij} = \frac{\partial W(\underset{\sim}{F})}{\partial F_{ij}} , \tag{2.9}$$

or, equivalently because of (2.7),

$$\tau_{ij} = \frac{1}{J}\frac{\partial W(\underset{\sim}{F})F_{jk}}{\partial F_{ik}} \tag{2.10}$$

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[1]The τ_{ij} represent forces per unit <u>deformed</u> area.

The displacement equations of equilibrium for a compressible material then follow from (2.9), (2.6) and the first of (2.3) as

$$\frac{\partial}{\partial x_j}\left[\frac{\partial W(\underset{\sim}{F})}{\partial F_{ij}}\right] \equiv c_{ijk\ell}(\underset{\sim}{F})u_{k,j\ell} = 0 \qquad \text{on } \mathcal{R} , \tag{2.11}$$

in which

$$c_{ijk\ell}(\underset{\sim}{F}) = \frac{\partial^2 W(\underset{\sim}{F})}{\partial F_{ij}\partial F_{k\ell}} \tag{2.12}$$ [1]

For an isotropic compressible elastic material, W depends on $\underset{\sim}{F}$ only through the invariants I_1, I_2 and J : $W = W(I_1, I_2, J)$, where

$$I_1 = \mathrm{Tr}\,\underset{\sim}{G} \ , \quad I_2 = 1/2\left[(\mathrm{Tr}\,\underset{\sim}{G})^2 - \mathrm{Tr}\,(\underset{\sim}{G}^2)\right] , \tag{2.13}$$

and Tr denotes the trace. In this case (2.9) specializes to

$$\sigma_{ij} = 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)F_{ij} - 2\frac{\partial W}{\partial I_2}F_{ik}C_{kj} + J\frac{\partial W}{\partial J}F^{-1}_{ji} , \tag{2.14}$$

where F^{-1}_{ij} is the i,j - element of the inverse $\underset{\sim}{F}^{-1}$ of $\underset{\sim}{F}$, and C_{ij} is given by the last of (2.3).

For an incompressible homogeneous elastic material, only those deformations which preserve volume locally are admissible, and (2.9), (2.10) are replaced respectively by

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[1] $W(\underset{\sim}{F})$ is assumed to be infinitely differentiable for every nonsingular tensor $\underset{\sim}{F}$.

$$\sigma_{ij} = \frac{\partial W(\underset{\sim}{F})}{\partial F_{ij}} - pF^{-1}_{ji} \ , \tag{2.15}$$

$$\tau_{ij} = \frac{\partial W(\underset{\sim}{F})}{\partial F_{ik}} F_{jk} - p\delta_{ij} \ , \tag{2.16}$$

where δ_{ij} is the Kronecker delta and p is an arbitrary hydrostatic pressure, assumed to be continuously differentiable on $\mathcal{R}$. The displacement equations of equilibrium for an incompressible material are now

$$c_{ijk\ell}(\underset{\sim}{F})u_{k,j\ell} - F^{-1}_{ji}p_{,j} = 0 \qquad \text{on } \mathcal{R} \ , \tag{2.17}$$

together with the constraint expressing incompressibility:

$$J = \det \underset{\sim}{F} = 1 \qquad \text{on } \mathcal{R} \ . \tag{2.18}$$

The functions $c_{ijk\ell}$ in (2.17) are again given by (2.12); (2.17) follows from (2.6), (2.15) and the fact that $F^{-1}_{ji,j} \equiv 0$ if $\det \underset{\sim}{F} = 1$. The system (2.17), (2.18) comprises four equations for the four unknowns u_i, p.

For an isotropic incompressible material, W depends on $\underset{\sim}{F}$ only through the invariants I_1, I_2 of (2.13), and (2.15) specializes to

$$\sigma_{ij} = 2\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right) F_{ij} - 2\frac{\partial W}{\partial I_2} F_{ik}C_{kj} - pF^{-1}_{ji} \ . \tag{2.19}$$

3. Weak solutions and equilibrium shocks

In place of the smoothness assumptions made in the preceding section, it will now be required that the displacement vector $\underset{\sim}{u}$ be merely continuous and piecewise continuously differentiable on $\mathcal{R}$, while the nominal

stress tensor $\underset{\sim}{\sigma}$ is piecewise continuous on $\mathcal{R}$. For incompressible materials, it is further assumed that the hydrostatic pressure p is piecewise continuous on $\mathcal{R}$. Thus $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ (and p, for incompressible materials) may suffer jump discontinuities across certain surfaces[1] in $\mathcal{R}$.

Enlarging the class of admissible stress and displacement fields in this way makes it necessary to generalize the sense in which certain of the field equations are to be satisfied. The tensors $\underset{\sim}{F}$, $\underset{\sim}{G}$ and $\underset{\sim}{C}$ whose components are defined in (2.3) are now piecewise continuous on $\mathcal{R}$, as is the Jacobian J of (2.4). Equation (2.5), expressing the local balance of mass, continues to have meaning under the present circumstances and is in fact equivalent to the integral condition expressing mass balance in the large. The local equations (2.6) of force equilibrium, however, must be replaced by the corresponding global conditions from which they were originally derived:

$$\int_S \sigma_{ij} N_j \, dA = 0 \qquad \text{for every closed regular surface } S \text{ lying in } \mathcal{R}. \tag{3.1}$$

Here $\underset{\sim}{N}$ is the unit outward normal on S. The integral version (3.1) of the force balance requirement remains meaningful for nominal stress fields $\underset{\sim}{\sigma}$ which are merely piecewise continuous on $\mathcal{R}$.

The components τ_{ij} of true stress are still to be calculated from σ_{ij} and F_{ij} according to (2.7). Regarded as functions on $\mathcal{R}^*$, the τ_{ij} are piecewise continuous. The symmetry condition $\tau_{ij} = \tau_{ji}$ continues to be equivalent to the global balance of moments under present circumstances; this imposes via (2.7) the restriction

[1] These surfaces are assumed to be regular in the sense of Kellogg. [12].

$$\sigma_{ik} F_{jk} = \sigma_{jk} F_{ik} \qquad \text{on } \mathcal{R} \,. \tag{3.2}$$ [1]

Under the smoothness assumptions now in force, $\underset{\sim}{u}$, $\underset{\sim}{F}$ and $\underset{\sim}{\sigma}$ are said to furnish a <u>weak solution</u> of the field equations for compressible materials if (2.9), the first of (2.3), (2.5), (3.1) and (3.2) hold. The inequality (2.4) is, of course, still to be imposed.

For incompressible materials, a weak solution is furnished by $\underset{\sim}{u}$, $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ and p provided the above equations hold, except that (2.9) is to be replaced by (2.15).

It is possible to enlarge still further the class of admissible fields $\underset{\sim}{u}$, $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ (and p, where appropriate). One such enlargement would permit unbounded displacements and/or stresses at isolated points or along curves[2] in $\mathcal{R}$. A second — and considerably more significant — weakening of the smoothness restrictions would permit discontinuities in the displacement vector $\underset{\sim}{u}$ itself. This particular relaxation of requirements would indeed be essential for the study of fracture. Neither of these generalizations will be considered here.

Suppose now that the deformation gradient tensor $\underset{\sim}{F}$ and the nominal stress field $\underset{\sim}{\sigma}$ associated with a weak solution (for either a compressible or an incompressible material) suffer jump discontinuities across a surface S in $\mathcal{R}$ but are continuously differentiable on either side of S in a neighborhood

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[1] It follows from (2.14) that (3.2) is automatically satisfied for an <u>isotropic</u> elastic material. In fact the same applies to all <u>objective</u> elastic materials: for these, W depends on $\underset{\sim}{F}$ only through $\underset{\sim}{C}$.

[2] This might be necessary, for example, to accommodate singularities at the edge of a terminating surface which carries discontinuites in $\underset{\sim}{F}$ or $\underset{\sim}{\sigma}$.

of a point P on S[1]. The global equilibrium conditions (3.1) then furnish two conclusions: first, by the familiar argument, $\underset{\sim}{\sigma}$ satisfies the local equilibrium conditions (2.6) on either side of S near P. Second, by applying (3.1) to a small sphere centered at P and using the fact that (2.6) hold on either side of S inside the sphere, one finds that the jumps in σ_{ij} must be such that

$$[\sigma_{ij}]_-^+ N_j = 0 \qquad \text{at } P \text{ on } S \,. \tag{3.3}$$

Here $[\sigma_{ij}]_-^+ = \overset{+}{\sigma}_{ij} - \bar{\sigma}_{ij}$, $\overset{+}{\sigma}_{ij}$ and $\bar{\sigma}_{ij}$ stand for the limiting values of σ_{ij} on S from the positive and negative sides of S, and the "positive" side of S is the side into which $\underset{\sim}{N}$ points. Equation (3.3) asserts that the <u>nominal</u> traction $\underset{\sim}{s}$ with components

$$s_i = \sigma_{ij} N_j \tag{3.4}$$

is necessarily continuous across S. It is possible to show that (3.3) is equivalent to the condition that the <u>true</u> traction $\underset{\sim}{t}$ with components $t_i = \tau_{ij}\; n_j$ be continuous across the deformation image S^* of S; here $\underset{\sim}{n}$ is the unit normal on S^*.

Let $\underset{\sim}{L}$ be any unit vector which is tangent to S at P. Since $\underset{\sim}{u}$ is continuous at P and continuously differentiable on either side of S, near enough to P it follows from the first of (2.3) that

$$[F_{ij}]_-^+ L_j = 0 \qquad \text{at } P \text{ on } S \tag{3.5}$$

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[1] S is assumed to have a continuously varying unit normal $\underset{\sim}{N}$ near P, and the derivatives $\sigma_{ij,k}$ are presumed to have finite limiting values from either side at all points of S which are sufficiently close to P.

A surface S in $\mathcal{R}$ carrying jump discontinuities in $\underset{\sim}{F}$ and $\underset{\sim}{\sigma}$ satisfying the jump conditions (3.3) and (3.5) is called an <u>equilibrium</u> <u>shock</u>. To distinguish S in $\mathcal{R}$ from its deformation image S^* in $\mathcal{R}^*$, it is convenient to refer to S as the <u>material</u> shock surface and to S^* as the <u>spatial</u> shock surface. It is of course the spatial shock S^* that one would actually "observe"; nevertheless, the pre-image S of S^* figures more prominently in the present analysis.

To investigate many of the questions pertaining to the local existence and local properties of equilibrium shocks, it is sufficient to consider the case in which S is a plane through P and $\underset{\sim}{u}$ is such that $\underset{\sim}{F}$ is constant on either side of S. Such <u>piecewise</u> <u>homogeneous</u> deformations and the associated equilibrium shocks have been studied in [3] for the case of plane strain in compressible materials.[1]

4. Energy considerations and dissipation.

The first objective of the present section is the derivation of a formula which reveals the effect on the mechanical energy balance in an elastic body of the presence of an equilibrium shock. This formula has connections with a conservation law arising in the elastostatics of homogeneous materials as well as with the theory of defects in an elastic solid; it will lead to a notion of dissipation for equilibrium shocks which is the principal concern of the present paper.

In order to obtain the energy formula, it is necessary to consider

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[1] It is proved in [3] that a necessary condition for the existence of an equilibrium shock in plane strain of a compressible material is that the (plane version) of the displacement equations of equilibrium (2.11) must suffer a loss of strong ellipticity at <u>some</u> deformation. A minor modification of the argument used in [3] shows that this result remains true for <u>three</u>-dimensional equilibrium shocks in compressible materials.

a quasi-static time-dependent family of equilibrium states. Thus let $\underset{\sim}{u}(\underset{\sim}{x}, t)$, $\underset{\sim}{F}(\underset{\sim}{x}, t)$, $\underset{\sim}{\sigma}(\underset{\sim}{x}, t)$ furnish, for each t in a suitable time interval, a weak solution of the field equations of equilibrium for a compressible material.[1] Note that no inertia effects are considered here: the time t plays only the role of a history parameter.[2] Suppose that this weak solution involves an equilibrium shock across the surface S_t^* in the deformed body $\mathcal{R}_t^*$ at time t, and let S_t be the pre-image in the undeformed body $\mathcal{R}$ of S_t^*. Thus S_t and S_t^* are the material and spatial shock surfaces, respectively.

Let $\underset{\sim}{V}(\underset{\sim}{x}, t)$ stand for the velocity vector[3] of a point on the moving surface S_t which at time t is located at $\underset{\sim}{x}$. It is now assumed that, at time t_o, the point $\underset{\sim}{x}_o$ on S_{t_o} is such that $\underset{\sim}{V}(\underset{\sim}{x}_o, t_o)$ neither vanishes nor is tangent to S_{t_o}. Thus S_{t_o} does not coincide locally with a "contact surface" — a surface across which there is no flow of particles.[4] It then follows that for times near enough to t_o, and for points $\underset{\sim}{x}$ on S_t which, at time t_o, were sufficiently close to $\underset{\sim}{x}_o$ on S_{t_o}, $\underset{\sim}{V}(\underset{\sim}{x}, t)$ will continue to have this property.

Now suppose that $\mathcal{D}$ is a regular subdomain of $\mathcal{R}$ which contains $\underset{\sim}{x}_o$. Then $\mathcal{D}$ will be intersected by S_t for all times in a suitable interval

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[1] $\underset{\sim}{u}$, $\underset{\sim}{F}$ and $\underset{\sim}{\sigma}$ are assumed to be continuously differentiable with respect to t for every $\underset{\sim}{x}$ in $\mathcal{R}$.

[2] If, for example, a cylindrical body is loaded on its ends to a state of uniaxial tension, t might be taken to be proportional to the given applied force (if the deformation is "load controlled") or to the given extension ("grip control"), as long as the force — or the extension — is a monotone function of real time.

[3] $\underset{\sim}{V}(\underset{\sim}{x}, t)$ is assumed to be continuous in $(\underset{\sim}{x}, t)$ for all $\underset{\sim}{x}$ on S_t and all times under consideration.

[4] See §52 of [7]. Thus S_t (and therefore S_t^*) in general consist of different material particles at different times.

including t_o. Let $\hat{S}_t$ denote the portion of S_t which lies in $\mathcal{D}$. For a sufficiently small domain $\mathcal{D}$ and for times t close enough to t_o, one may choose the unit normal vector $\underset{\sim}{N}$ on $\hat{S}_t$ so that $\underset{\sim}{V} \cdot \underset{\sim}{N} > 0$, as will now be assumed. The positive side of $\hat{S}_t$ is thus the side into which $\underset{\sim}{V}$ (and therefore $\underset{\sim}{N}$) points. Since, to an observer ~~fixed on~~ $\hat{S}_t$, particles appear to cross $\hat{S}_t$ from the positive to the negative side, it is natural to refer to the positive side as "upstream", the negative as "downstream". Let $\overset{+}{\mathcal{D}}_t$, $\overset{-}{\mathcal{D}}_t$ be the open domains[1] into which $\mathcal{D}$ is divided by $\hat{S}_t$ (see Fig. 1).

The total energy stored at a time t close enough to t_o in that portion of the deformed body which occupies the domain $\mathcal{D}_t^*$ — the image of $\mathcal{D}$ under the deformation at time t — is given by

$$U(t) = \int_{D} W(\underset{\sim}{F}(\underset{\sim}{x}, t))\, dV \quad . \tag{4.1}$$

The first objective is to obtain a formula for the derivative $\dot{U}(t)$. Clearly

$$U(t) = \int_{\overset{+}{\mathcal{D}}_t} W(\underset{\sim}{F}(\underset{\sim}{x}, t))\, dV + \int_{\overset{-}{\mathcal{D}}_t} W(\underset{\sim}{F}(\underset{\sim}{x}, t))\, dV \quad , \tag{4.2}$$

and so to find $\dot{U}(t)$ it is necessary to calculate the derivatives of the integrals in (4.2), in which the integrands and the domains of integration vary with time. Making use of a standard formula for this purpose,[2] one obtains

$$\dot{U}(t) = \int_{\overset{+}{\mathcal{D}}_t} \frac{\partial W}{\partial F_{ij}} \dot{F}_{ij} dV - \int_{\hat{S}_t} \overset{+}{W} \underset{\sim}{V} \cdot \underset{\sim}{N} dA + \int_{\overset{-}{\mathcal{D}}_t} \frac{\partial W}{\partial F_{ij}} \dot{F}_{ij} dV + \int_{\hat{S}_t} \overset{-}{W} \underset{\sim}{V} \cdot \underset{\sim}{N} dA \quad . \tag{4.3}$$

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[1] Note that, while $\overset{+}{\mathcal{D}}_t$ and $\overset{-}{\mathcal{D}}_t$ depend on t, $\mathcal{D}$ does not.

[2] See [13], Chapter 3, Section 6.

In this formula,

$$\dot{F}_{ij} = \dot{F}_{ij}(\underset{\sim}{x}, t) = \frac{\partial}{\partial t} F_{ij}(\underset{\sim}{x}, t) \quad , \tag{4.4}$$

and $\overset{+}{W}$, $\overset{-}{W}$ represent the limiting values of W on $\hat{S}_t$ from the upstream and downstream sides, respectively.

Let

$$\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x}, t) = \frac{\partial}{\partial t} \underset{\sim}{u}(\underset{\sim}{x}, t) \tag{4.5}$$

stand for the velocity of the particle which at time t is located at $\underset{\sim}{y}(\underset{\sim}{x}, t)$. Then from (2.3), (4.4) and (4.5)

$$\dot{F}_{ij}(\underset{\sim}{x}, t) = v_{i,j}(\underset{\sim}{x}, t) \quad . \tag{4.6}$$

The volume integrals in (4.3) can then be calculated as follows: using (2.9), (4.6), (2.6), and the divergence theorem,

$$\int_{\overset{\pm}{\mathcal{D}}_t} \frac{\partial W}{\partial F_{ij}} \dot{F}_{ij} dV = \int_{\overset{\pm}{\mathcal{D}}_t} \sigma_{ij} v_{i,j} dV = \int_{\partial\overset{\pm}{\mathcal{D}}_t} \sigma_{ij} \nu_j v_i dA \tag{4.7}$$

where $\underset{\sim}{\nu}$ is the unit outward normal on the boundary $\partial\overset{\pm}{\mathcal{D}}_t$ of $\overset{\pm}{\mathcal{D}}$. Substituting from (4.7) into (4.3) and observing that, on that portion of $\partial\overset{\pm}{\mathcal{D}}$ which coincides with $\hat{S}_t$, the normal $\underset{\sim}{\nu} = \mp\underset{\sim}{N}$, one finds

$$\dot{U}(t) = \int_{\partial\mathcal{D}} \sigma_{ij} \nu_j v_i dA - \int_{\hat{S}_t} (\overset{+}{\sigma}_{ij} N_j \overset{+}{v}_i + \overset{+}{W} N_i V_i) dA + \int_{\hat{S}_t} (\bar{\sigma}_{ij} N_j \bar{v}_i + \bar{W} N_i V_i) dA \quad , \tag{4.8}$$

where once again the superscripts + and - on σ_{ij} and v_i refer to the

appropriate limiting values of these quantities. To this point no use has been made of the jump conditions prevailing across an equilibrium shock. By differentiating with respect to t the equation which expresses the equality of the limiting values of $\underset{\sim}{u}$ on S_t, one finds that the jump in particle velocity $\underset{\sim}{v}$ satisfies

$$\left[v_i\right]_-^+ = -\left[F_{ik}\right]_-^+ V_k \tag{4.9}$$

in terms of the components V_k of the velocity of points on the (material) shock surface S_t and the jump in the components F_{ik} of the deformation gradient tensor $\underset{\sim}{F}$. When (4.9) and the nominal traction continuity requirement (3.3) are used in (4.8), it reduces to

$$\dot{U}(t) = \int_{\partial \mathcal{D}} s_i v_i dA - \int_{\hat{S}_t} \left[P_{ij}\right]_-^+ N_j V_i dA \ , \tag{4.10}$$

where

$$P_{ij} = W\delta_{ij} - F_{ki}\sigma_{kj} \ . \tag{4.11}$$

The P_{ij} are the components of the energy-momentum tensor $\underset{\sim}{P} = W\underset{\sim}{1} - \underset{\sim}{F}^T\underset{\sim}{\sigma}$, whose role in the theory of defects in solids has been explored extensively by Eshelby [14, 15, 16]. Indeed, (4.10) suggests that $[\underset{\sim}{P}]\underset{\sim}{N}$ may be viewed as a "force per unit area" associated with the material shock surface S_t. In this interpretation, the role of $\underset{\sim}{P}$ is precisely that discussed by Eshelby in § 6 of [16] in connection with the "force on an interface".

In general the "force on a defect" has been shown [14, 15, 16] to

have components

$$J_i = \oint_{\Gamma} P_{ij} N_j dA \quad , \tag{4.12}$$

where Γ is a closed surface surrounding the defect. If there are no defects inside Γ, so that $\underset{\sim}{u}$ is twice continuously differentiable there,

$$J_i = \oint_{\Gamma} P_{ij} N_j dA = 0 \quad ; \tag{4.13}$$ [1]

(4.13) represents a conservation law for compressible homogeneous elastic materials which was independently discovered by Rice [17] and exploited by him — and subsequently others — in connection with crack problems in elasticity. The energy-momentum tensor $\underset{\sim}{P}$ plays a prominent part in the dissipation condition to be derived below. This would seem to be related to the fact that, as emphasized by Lax [11], "entropy conditions" appropriate for shocks associated with the initial value problem for a quasilinear system are connected with additional conservation laws implied by the system.

An alternative and more convenient form of the energy formula (4.10) is obtained by noting that, if $\underset{\sim}{L}$ is any vector tangent to S_t, (4.11) gives

$$P_{ij} N_j L_i = - F_{ki} \sigma_{kj} L_i N_j \quad , \tag{4.14}$$

---

[1] Equation (4.13) is easily verified directly with the help of the divergence theorem and the field equations of Section 2 appropriate to compressible materials.

so that by (3. 5), (3. 3),

$$\left[P_{ij}\right]_-^+ N_j L_i = - \left[F_{mi}\right]_-^+ L_i \left[\sigma_{mj}\right]_-^+ N_j = 0 \ . \tag{4.15}$$

Thus the vector $[\underset{\sim}{P}]_-^+ \underset{\sim}{N}$ is perpendicular to S_t ; it follows that (4. 10) may be written in the form

$$\dot{U}(t) = \int_{\partial\mathcal{D}} \underset{\sim}{s} \cdot \underset{\sim}{v} dA + \int_{\hat{S}_t} [H]_-^+ \underset{\sim}{V} \cdot \underset{\sim}{N} dA \ , \tag{4.16}$$

where

$$H = H(\underset{\sim}{F}, \underset{\sim}{N}) = - P_{ij} N_i N_j = - W + F_{mi} \sigma_{mj} N_i N_j \ . \tag{4.17}$$

The identity (4. 16) is the energy formula which furnishes the basis for the notion of a dissipative shock to be introduced below. One first notes that $[H]_-^+ = 0$ if no shock is present, so that (4. 16) then reduces to the statement that the rate of change of energy stored in any portion of the elastic body is equal to the rate at which work is done by the forces external to that portion. If an equilibrium shock is present, it is natural to require that the second term on the right in (4. 16) be nonpositive, corresponding to the idea that the shock should be dissipative in the sense that it tend to diminish (or at least not increase) the stored energy in the body. One is accordingly led to require that, at each instant t ,

$$\int_{\hat{S}_t} [H]_-^+ \underset{\sim}{V} \cdot \underset{\sim}{N} dA \le 0 \tag{4.18}$$

for every subdomain $\mathcal{D}$ of $\mathcal{R}$ which is intersected by S_t and which is small enough to permit $\underset{\sim}{N}$ to be chosen so that $\underset{\sim}{V} \cdot \underset{\sim}{N} > 0$ on $\hat{S}_t = S_t \cap \mathcal{D}$.

Suppose (4.18) holds, and let $\underset{\sim}{x}$ be a non-contact point on S_t, i.e, a point at which $\underset{\sim}{V}$ is neither zero nor tangent to S_t. Since $\mathcal{D}$ may be chosen as the interior of an arbitrarily small sphere centered at $\underset{\sim}{x}$, it follows that

$$[H]_-^+ \leq 0 \text{ at all non-contact points of } S_t \, , \tag{4.19}$$

provided one interprets the positive side of S_t as the side into which $\underset{\sim}{V}$ points.

According to (4.17), the scalar H in the dissipation inequality (4.19) depends not only on the particular particle concerned (through $\underset{\sim}{F}$) but on the orientation of the shock (through $\underset{\sim}{N}$) as well. The inequality (4.19) may thus be interpreted as requiring that the value of H associated with a given particle and a given $\underset{\sim}{N}$ shall not decrease as the particle crosses the shock, as is true of the entropy in gas-dynamic shocks.

In what follows, (4.19) will be required to hold at all times t.

The version of (4.19) appropriate to equilibrium shocks in piecewise homogeneous plane deformations of a compressible elastic material was derived in [3] and applied there to certain specific strain energy densities.

For incompressible materials, the argument used to obtain (4.16) can be repeated as above, except that (4.7) must be replaced by

$$\int_{\mathcal{D}^{\pm}} \frac{\partial W}{\partial F_{ij}} \dot{F}_{ij} dV = \int_{\mathcal{D}^{\pm}} (\sigma_{ij} + pF_{ji}^{-1}) v_{i,j} dV = \int_{\partial \mathcal{D}^{\pm}} \sigma_{ij} \nu_j v_i dA + \int_{\mathcal{D}^{\pm}} p \frac{\partial v_i}{\partial y_i} dV \, . \tag{4.20}$$

Here (2.15) has been used, as well as the relation

$$F^{-1}_{ji} v_{i,j} = \partial v_i / \partial y_i \tag{4.21}$$

in which, on the left, v_i is regarded as a function of $\underset{\sim}{x}$ and t, while on the right, as a function of $\underset{\sim}{y}$ and t. The incompressibility condition (2.18), however, implies that $\partial v_i / \partial y_i = 0$, so that the last integral on the right in (4.20) vanishes, reducing (4.20) to the same result obtained in (4.7). It then follows that (4.16) holds also for incompressible materials with H again given by (4.17). Thus the inequality (4.19) is the natural dissipation requirement to impose on equilibrium shocks in incompressible materials as well.

5. Compressible materials: weak shocks in plane strain.

If the region $\mathcal{R}$ occupied by the undeformed body is cylindrical with generators parallel to the x_3 - axis, the deformation (2.1) corresponds to plane strain if $u_3 \equiv 0$ and[1] $u_\alpha = u_\alpha(x_1, x_2)$. For plane strain, $\underset{\sim}{F}$, $\underset{\sim}{G}$ and $\underset{\sim}{C}$ of (2.2) are functions of x_1, x_2 only and are such that $F_{\alpha 3} = F_{3\alpha} = C_{3\alpha} = C_{\alpha 3} = G_{3\alpha} = G_{\alpha 3} = 0$, while $F_{33} = C_{33} = G_{33} = 1$.

Not all elastic materials can sustain general states of plane strain in the absence of body forces. A sufficient condition that a material shall have this property is that, under the above circumstances, $\underset{\sim}{\sigma}$ as determined by (2.9) shall be such that $\sigma_{3\alpha} = \sigma_{\alpha 3} = 0$. This is assumed hereafter in the present section. An isotropic material always fulfills this condition.

For plane strain the displacement equations of equilibrium (2.11)

---

[1] Greek subscripts have the range 1,2.

in the compressible case are readily shown to reduce to

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F})u_{\gamma,\beta\delta} = 0 \qquad \text{on } \mathcal{R}_o \ , \tag{5.1}$$

where $\mathcal{R}_o$ is the cross-section of the cylinder $\mathcal{R}$ which lies in the plane $x_3 = 0$, and

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \frac{\partial^2 W(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\beta\gamma}} \ . \tag{5.2}$$

If the deformation is plane, one shows easily from (2.13) and (2.4) that

$$I_1 = 1 + I \ , \ I_2 = I + J^2 \ , \ I = G_{\alpha\alpha} = F_{\alpha\beta}F_{\alpha\beta} \ , \tag{5.3}$$

so that for isotropic materials, $W = W(I_1, I_2, J) = W(I, J)$. It can further be shown[1] that

$$c_{\alpha\beta\gamma\delta} = 4W_{II}F_{\alpha\beta}F_{\gamma\delta} + 2JW_{IJ}\left(F^{-1}_{\beta\alpha}F_{\gamma\delta} + F_{\alpha\beta}F^{-1}_{\delta\gamma}\right) + J^2F^{-1}_{\beta\alpha}F^{-1}_{\delta\gamma}W_{JJ} + 2W_I\delta_{\alpha\gamma}\delta_{\beta\delta}$$
$$+ JW_J\left(F^{-1}_{\beta\alpha}F^{-1}_{\delta\gamma} - F^{-1}_{\delta\alpha}F^{-1}_{\beta\gamma}\right) \tag{5.4}$$

for isotropic materials; here subscripts I and J indicate partial differentiation of $W(I, J)$.

In plane strain, both the material and the spatial equilibrium shock surfaces are cylindrical, and a material shock surface intersects the cross-section $\mathcal{R}_o$ in a curve[2] C. The jump conditions (3.3) and

---

[1]See [2], Eq. (1.17).

[2] C is assumed to have a continuously varying normal.

(3.5) become

$$\left[\sigma_{\alpha\beta}\right]_{-}^{+} N_{\beta} = 0 \qquad \text{on } C \tag{5.5}$$

and

$$\left[F_{\alpha\beta}\right]_{-}^{+} L_{\beta} = 0 \qquad \text{on } C \ , \tag{5.6}$$

where $\underset{\sim}{N}$ and $\underset{\sim}{L}$ are the unit normal and unit tangent vectors on C, respectively, and the positive side of C is the side into which $\underset{\sim}{N}$ is directed

In general the various physical and geometrical quantities $\overset{\pm}{F}_{\alpha\beta}$, $\overset{\pm}{\sigma}_{\alpha\beta}$, N_{α}, L_{α} will vary with position along C. Moreover, if the shock is regarded as a member of a quasi-static time-dependent family of equilibrium shocks in the sense of the preceding section, the values of these quantities will also vary with time. In the present section attention is confined to the local structure of the various fields at a fixed point on C and at a fixed instant of time. Suppose at such a fixed point and fixed time, the values of $\overset{+}{F}_{\alpha\beta}$ (and therefore $\overset{+}{\sigma}_{\alpha\beta}$) are regarded as given. The jump conditions (5.5) and (5.6) then restrict the possible local values of $\bar{F}_{\alpha\beta}$, $\bar{\sigma}_{\alpha\beta}$, L_{α} and N_{α} associated with the equilibrium shock. Since

$$\sigma_{\alpha\beta} = \frac{\partial W(\underset{\sim}{F})}{\partial F_{\alpha\beta}} \ , \tag{5.7}$$

and in view of the fact that $\underset{\sim}{N}$ and $\underset{\sim}{L}$ can be expressed solely in terms of the local angle of inclination Φ of $\underset{\sim}{L}$ with respect to the x_1 - axis, the jump conditions (5.5) and (5.6) may be regarded as four equations for the five unknowns $\bar{F}_{\alpha\beta}$ and Φ. One would thus expect to find a one-parameter family of "solutions" $\bar{F}_{\alpha\beta}$, Φ of (5.5), (5.6) for given $\overset{+}{F}_{\alpha\beta}$. It is convenient to choose as the parameter ϵ indexing this family the relative area-

change across the shock, so that, in view of (2.5),

$$\epsilon = \frac{\overset{+}{J} - \overset{-}{J}}{\overset{+}{J}} = \frac{\overset{+}{\rho} - \overset{-}{\rho}}{\overset{-}{\rho}} \quad . \tag{5.8}$$

One now regards $\overset{-}{F}_{\alpha\beta} = \overset{-}{F}_{\alpha\beta}(\epsilon)$, $\Phi = \Phi(\epsilon)$, and therefore $L_\alpha = L_\alpha(\epsilon)$ and $N_\alpha = N_\alpha(\epsilon)$, as functions of ϵ.

If in fact no shock is present, so that $\overset{+}{J} = \overset{-}{J}$, (5.8) shows that $\epsilon = 0$. A <u>weak</u> equilibrium shock is one for which $|\overset{-}{F}_{\alpha\beta}(\epsilon) - \overset{+}{F}_{\alpha\beta}|$, and therefore $|\epsilon|$, are small compared to unity. It is now assumed that (5.5) and (5.6), with the aid of (5.7), determine, for a given $\overset{+}{F}_{\alpha\beta}$, a smooth[1] family $\overset{-}{F}_{\alpha\beta}(\epsilon)$, $N_\alpha(\epsilon)$, $L_\alpha(\epsilon)$ near $\epsilon = 0$ which is such that

$$\overset{-}{F}_{\alpha\beta}(0) = \overset{+}{F}_{\alpha\beta} \quad . \tag{5.9}$$

The objective of the analysis in the remainder of this section is the determination of the weak-shock approximation to the local value of the jump $[H]_-^+$ in the function H of (4.17) for plane strain of a compressible material. This necessitates a study of the corresponding approximations[2] to $\overset{-}{F}_{\alpha\beta}(\epsilon)$ and $N_\alpha(\epsilon)$.

---

[1]Specifically $\overset{-}{F}_{\alpha\beta}(\epsilon)$ and $\Phi(\epsilon)$ must be four times and three times continuously differentiable, respectively.

[2]Weak shocks are studied in detail in [3], but more refined calculations than those given in [3] are required to find the weak shock approximation to $[H]_-^+$.

To begin, one observes that the kinematical jump condition (5.6) immediately leads to

$$\bar{F}_{\alpha\beta}(\epsilon) = \overset{+}{F}_{\alpha\beta} + g_\alpha(\epsilon) N_\beta(\epsilon) \tag{5.10}$$ [1]

where $\underset{\sim}{g}(\epsilon)$ is an as yet undetermined, four times continuously differentiable function which, by (5.9), must satisfy

$$g_\alpha(0) = 0 \quad . \tag{5.11}$$

The vector $\underset{\sim}{g}(\epsilon)$ represents the jump across the shock of the derivative of the displacement vector normal to the shock.

In order to analyze the traction continuity condition (5.5), set

$$\Delta_\alpha(\epsilon) = \left[\frac{\partial W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \frac{\partial W(\bar{\underset{\sim}{F}}(\epsilon))}{\partial F_{\alpha\beta}} \right] N_\beta(\epsilon) \quad , \tag{5.12}$$

and observe that, for all sufficiently small $|\epsilon|$,

$$\Delta_\alpha(\epsilon) = 0 \tag{5.13}$$

But (5.12) and (5.2) show that

$$\Delta'_\alpha(\epsilon) = \left[\frac{\partial W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \frac{\partial W(\bar{\underset{\sim}{F}}(\epsilon))}{\partial F_{\alpha\beta}} \right] N'_\beta(\epsilon)$$
$$- c_{\alpha\beta\gamma\delta}(\bar{\underset{\sim}{F}}(\epsilon)) \bar{F}'_{\gamma\delta}(\epsilon) N_\beta(\epsilon) \quad , \tag{5.14}$$

---

[1] Equation (5.10), which is not limited to weak shocks, leads to the geometrical interpretation of the local deformation associated with equilibrium shocks. Although this interpretation will not be discussed here, it is explained fully in [3].

where the prime indicates differentiation with respect to ϵ. From (5.10), (5.11),

$$\bar{F}'_{\alpha\beta}(\epsilon) = g'_{\alpha}(\epsilon)N_{\beta}(\epsilon) + g_{\alpha}(\epsilon)N'_{\beta}(\epsilon) \ , \quad \bar{F}'_{\alpha\beta}(0) = g'_{\alpha}(0)N_{\beta}(0) \ . \tag{5.15}$$

Since according to (5.13), $\Delta'_{\alpha}(0) = 0$, one concludes from (5.14), (5.15) and (5.9) that $g'_{\gamma}(0)$ must satisfy

$$Q_{\alpha\gamma}(\underset{\sim}{N}(0), \overset{+}{\underset{\sim}{F}})g'_{\gamma}(0) = 0 \tag{5.16}$$

where

$$Q_{\alpha\gamma}(\underset{\sim}{N}, \underset{\sim}{F}) = c_{\alpha\beta\gamma\delta}(\underset{\sim}{F})N_{\beta}N_{\delta} \tag{5.17}$$

for any unit vector $\underset{\sim}{N}$ and nonsingular tensor $\underset{\sim}{F}$. It can be shown from (5.8) and (5.10) that $\underset{\sim}{g}(\epsilon)$ satisfies

$$\underset{\sim}{g}(\epsilon)\cdot\left(\overset{+}{\underset{\sim}{F}}{}^{-1}\right)^{T}\underset{\sim}{N}(\epsilon) = -\epsilon \ ; \tag{5.18}$$

differentiating (5.18) with respect to ϵ, setting $\epsilon = 0$ and using (5.11) shows that

$$\underset{\sim}{g}'(0)\cdot\left(\overset{+}{\underset{\sim}{F}}{}^{-1}\right)^{T}\underset{\sim}{N}(0) = -1 \ , \tag{5.19}$$

so that $\underset{\sim}{g}'(0) \neq \underset{\sim}{0}$. The existence of a nontrivial vector $\underset{\sim}{g}'(0)$ satisfying (5.16) requires that

$$\det \underset{\sim}{Q}(\underset{\sim}{N}(0), \overset{+}{\underset{\sim}{F}}) = 0 \ . \tag{5.20}$$

The condition (5.20) is necessary and sufficient for the failure of ellipticity

of the plane displacement equations of equilibrium[1] (5.1) at a deformation whose local gradient is $\overset{+}{\underset{\sim}{F}}$. If, for a given $\overset{+}{\underset{\sim}{F}}$, $\underset{\sim}{N}(0)$ satisfies (5.20), then $\underset{\sim}{N}(0)$ is normal to a <u>characteristic</u> <u>curve</u> of the system (5.1). For any such $\underset{\sim}{N}(0)$, one can show that (5.16) and the normalization condition (5.19) determine $\underset{\sim}{g}'(0)$ uniquely, provided that the symmetric tensor $\underset{\sim}{Q}$ is not the null tensor.[2]

It is unfortunately necessary to compute $\Delta''_\alpha(0)$. Differentiating (5.14) and setting $\epsilon = 0$ gives, with the aid of (5.9) and (5.15),

$$
\begin{aligned}
\Delta''_\alpha(0) = &- c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\Big\{2g'_\gamma(0)\left[N_\delta(0)N'_\beta(0) + N'_\delta(0)N_\beta(0)\right] \\
&\qquad + g''_\gamma(0)N_\delta(0)N_\beta(0)\Big\} \\
&- d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})g'_\gamma(0)g'_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \ ,
\end{aligned}
\tag{5.21}
$$

where

$$
d_{\alpha\beta\gamma\delta\lambda\mu}(\underset{\sim}{F}) = \frac{\partial}{\partial F_{\lambda\mu}} c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \frac{\partial^3 W(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}\partial F_{\lambda\mu}} \ . \tag{5.22}
$$

Since (5.13) holds for all sufficiently small $|\epsilon|$, it follows that $\Delta''_\alpha(0) = 0$ and therefore that $g'_\alpha(0)\Delta''_\alpha(0) = 0$. This, together with (5.16) and the symmetry $c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta}$ apparent from (5.2), lead to

$$
\begin{aligned}
&c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})g'_\alpha(0)g'_\gamma(0)N_\beta(0)N'_\delta(0) \\
&= -\frac{1}{4} d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})g'_\alpha(0)g'_\gamma(0)g'_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \ .
\end{aligned}
\tag{5.23}
$$

---

[1] See Section 1 of [1]

[2] The case $\underset{\sim}{Q} \equiv \underset{\sim}{0}$ is excluded from consideration.

This formula will be used later.

To compute $[H]_-^+$, set

$$h(\epsilon) = [H]_-^+ = H(\overset{+}{\underset{\sim}{F}}, \underset{\sim}{N}(\epsilon)) - H(\overset{-}{\underset{\sim}{F}}(\epsilon), \underset{\sim}{N}(\epsilon)) \quad . \tag{5.24}$$

Invoking the special features of $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ and $\underset{\sim}{N}$ appropriate to plane strain and using the definition (4.17) as well as (5.7), one obtains

$$\begin{aligned} h(\epsilon) = {} & W(\overset{-}{\underset{\sim}{F}}(\epsilon)) - W(\overset{+}{\underset{\sim}{F}}) \\ & + \frac{\partial W}{\partial F_{\gamma\beta}}(\overset{+}{\underset{\sim}{F}}) N_\beta(\epsilon) \overset{+}{F}_{\gamma\alpha} N_\alpha(\epsilon) \\ & - \frac{\partial W}{\partial F_{\gamma\beta}}(\overset{-}{\underset{\sim}{F}}(\epsilon)) N_\beta(\epsilon) \overset{-}{F}_{\gamma\alpha}(\epsilon) N_\alpha(\epsilon) \quad . \end{aligned} \tag{5.25}$$

If one subsitutes from (5.10) for $\overset{-}{F}_{\gamma\alpha}(\epsilon)$ where it appears explicitly in (5.25), one finds

$$\begin{aligned} h(\epsilon) = {} & W(\overset{-}{\underset{\sim}{F}}(\epsilon)) - W(\overset{+}{\underset{\sim}{F}}) - \frac{\partial W}{\partial F_{\gamma\beta}}(\overset{-}{\underset{\sim}{F}}(\epsilon)) g_\gamma(\epsilon) N_\beta(\epsilon) \\ & + \overset{+}{F}_{\gamma\alpha} N_\alpha(\epsilon) \left[\frac{\partial W}{\partial F_{\gamma\beta}}(\overset{+}{\underset{\sim}{F}}) N_\beta(\epsilon) - \frac{\partial W}{\partial F_{\gamma\beta}}(\overset{-}{\underset{\sim}{F}}(\epsilon)) N_\beta(\epsilon) \right. \quad . \end{aligned} \tag{5.26}$$

The traction continuity condition (5.5), together with (5.7), shows that the quantity in brackets in (5.26) vanishes, so that

$$h(\epsilon) = W(\overset{-}{\underset{\sim}{F}}(\epsilon)) - W(\overset{+}{\underset{\sim}{F}}) - \frac{\partial W}{\partial F_{\gamma\beta}}(\overset{-}{\underset{\sim}{F}}(\epsilon)) g_\gamma(\epsilon) N_\beta(\epsilon) \quad . \tag{5.27}$$

Obviously $h(0) = 0$. To determine the small $-|\epsilon|$ approximation to $h(\epsilon)$, one must compute some derivatives of h at $\epsilon = 0$. Differentiating (5.27),

using the first of (5.15) and referring to (5.2), one gets

$$h'(\epsilon) = c_{\alpha\beta\gamma\delta}(\bar{\underset{\sim}{F}}(\epsilon))g_\alpha(\epsilon)N_\beta(\epsilon)\bar{F}'_{\gamma\delta}(\epsilon) \ . \tag{5.28}$$

In view of (5.11),

$$h'(0) = 0 \ . \tag{5.29}$$

Differentiate (5.28) and use (5.22) and the first of (5.15) to find

$$\begin{aligned} h''(\epsilon) &= d_{\alpha\beta\gamma\delta\lambda\mu}(\bar{\underset{\sim}{F}}(\epsilon))g_\alpha(\epsilon)N_\beta(\epsilon)\bar{F}'_{\gamma\delta}(\epsilon)\bar{F}'_{\lambda\mu}(\epsilon) \\ &\quad + c_{\alpha\beta\gamma\delta}(\bar{\underset{\sim}{F}}(\epsilon))\left[\bar{F}'_{\alpha\beta}(\epsilon)\bar{F}'_{\gamma\delta}(\epsilon) + g_\alpha(\epsilon)N_\beta(\epsilon)\bar{F}''_{\gamma\delta}(\epsilon)\right] \ . \end{aligned} \tag{5.30}$$

From (5.11), (5.9) and (5.15) it then follows that

$$h''(0) = c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})g'_\alpha(0)g'_\gamma(0)N_\beta(0)N_\delta(0) \tag{5.31}$$

and hence from (5.16) that

$$h''(0) = 0 \ . \tag{5.32}$$

Finally, differentiate (5.30) and set $\epsilon = 0$, making use of (5.11), (5.9), (5.15), (5.16) and the symmetry $c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta}$. This yields

$$\begin{aligned} h'''(0) &= 2d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})g'_\alpha(0)g'_\gamma(0)g'_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \\ &\quad + 6c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})g'_\alpha(0)g'_\gamma(0)N_\beta(0)N'_\delta(0) \ . \end{aligned} \tag{5.33}$$

Appealing to (5.23), one arrives at

$$h'''(0) = \frac{1}{2} d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}}) g'_\alpha(0) g'_\gamma(0) g'_\delta(0) N_\beta(0) N_\delta(0) N_\mu(0) \quad . \tag{5.34}$$

Since h vanishes to second order at $\epsilon = 0$, one has

$$h(\epsilon) = \frac{1}{6} h'''(0)\epsilon^3 + O(\epsilon^4) \; , \; \epsilon \to 0 \quad , \tag{5.35}$$

and hence, using (5.24), the weak-shock result

$$[H]^+_- = D\epsilon^3 + O(\epsilon^4) \quad \text{as} \quad \epsilon \to 0 \quad , \tag{5.36}$$

where, by (5.33), (5.22)

$$D = \frac{1}{12} \frac{\partial^3 W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta} \partial F_{\gamma\delta} \partial F_{\lambda\mu}} g'_\alpha(0) g'_\gamma(0) g'_\lambda(0) N_\beta(0) N_\delta(0) N_\mu(0) \quad . \tag{5.37}$$

The formulas (5.36), (5.37) comprise the main result of the present section. They express the jump $[H]^+_-$ in terms of the third gradients of the strain energy density W, the weak-shock limiting normal $\underset{\sim}{N}(0)$, and the weak-shock approximation $\underset{\sim}{g}'(0)\epsilon$ to the jump $\underset{\sim}{g}(\epsilon)$ in the normal derivative of the displacement vector across the shock; $\underset{\sim}{g}'(0)$ is a nontrivial solution of (5.16) which also satisfies (5.19).

The fact that, for weak shocks, the jump $[H]^+_-$ is of the third order in the shock strength ϵ is analogous to the result in gas dynamics that the entropy jump across a weak shock is of third order in the corresponding shock strength parameter.[1]

---

[1] See p. 143 of [7].

Consider a quasi-static time-dependent family of plane strain equilibrium shocks in a given material, and let $\underset{\sim}{x}$ be the position vector of a point on the material shock surface S_t at a fixed instant t. Let $\underset{\sim}{V}(\underset{\sim}{x}, t)$ be the velocity of the point $\underset{\sim}{x}$ at the instant t, $\overset{+}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}(\underset{\sim}{x}, t)$ the limiting value of $\underset{\sim}{F}$ at $\underset{\sim}{x}, t$ from the upstream side. Let $\underset{\sim}{N}(0)$ satisfy (5.20) and be such that $\underset{\sim}{V}(\underset{\sim}{x}, t) \cdot \underset{\sim}{N}(0)$ is positive.[1] Let $\underset{\sim}{g}'(0)$ be the vector determined uniquely by (5.16), (5.19). Reference to (5.37) shows that the value of D at the point $\underset{\sim}{x}$ and the instant t is then fully determined by $\overset{+}{\underset{\sim}{F}}$ and $\underset{\sim}{N}(0)$. If the shock is sufficiently weak at $\underset{\sim}{x}, t$, (5.36) shows that the <u>sign</u> of the shock strength $\epsilon = \epsilon(\underset{\sim}{x}, t)$ is determined[2] by the dissipation inequality (4.19).[3] This in turn determines the sign of the jumps across the shock in other field quantities such as the density (see (5.8)).

If the compressible material at hand is isotropic, it is possible to use (5.37) (5.22), (5.4) and (5.19) to show that D assumes the following special form:

$$
\begin{aligned}
D = & -\frac{1}{12}\overset{+}{J}{}^3 W_{JJJ} - \frac{1}{2}\overset{+}{J} W_{IJ} + \left(W_{II} + \frac{1}{2}\overset{+}{J}{}^2 W_{IJJ}\right)\zeta \\
& - \overset{+}{J} W_{IIJ}\zeta^2 + \frac{2}{3} W_{III}\zeta^3 , \qquad (5.38)
\end{aligned}
$$

[1] If $\underset{\sim}{N}(0)$ satisfies (5.20), so does $-\underset{\sim}{N}(0)$ (see (5.17)). Thus if $\underset{\sim}{x}$ is a non-contact point of S_t (as is assumed here), then a vector $\underset{\sim}{N}(0)$ satisfying (5.20) may be assumed to satisfy $\underset{\sim}{V}(\underset{\sim}{x}, t) \cdot \underset{\sim}{N}(0) > 0$ without loss of generality.

[2] Except in the special case when $D = 0$; this eventuality will not be considered here.

[3] See [3] for details in special cases.

where

$$\zeta = \overset{+}{F}_{\alpha\beta} g'_{\alpha}(0) N_{\beta}(0) \quad , \tag{5.39}$$

and the derivatives of W appearing in (5.38) are to be evaluated at the values $\overset{+}{I}$ and $\overset{+}{J}$ appropriate to $\overset{+}{\underset{\sim}{F}}$.

6. Incompressible materials: equilibrium shocks in anti-plane strain.

A simpler class of deformations for the cylindrical body of the preceding section is that of anti-plane strain, in which the displacement vector satisfies

$$u_1 \equiv u_2 \equiv 0 \quad , \quad u_3 = u\left(x_1, x_2\right) \quad . \tag{6.1}$$

A particle is therefore displaced only axially, and the amount of displacement depends only upon the position of the particle in its cross-section. It follows from the first of (2.3) that the deformation gradient tensor $\underset{\sim}{F}$ associated with the displacement field (6.1) has components

$$F_{\alpha\beta} = \delta_{\alpha\beta} , \quad F_{\alpha 3} = 0 , \quad F_{3\alpha} = u,_{\alpha} , \quad F_{33} = 1 \quad . \tag{6.2}$$

From (6.2) and (2.13) one obtains

$$I_1 = I_2 = 3 + |\underset{\sim}{\nabla} u|^2 , \quad J = \det \underset{\sim}{F} = 1 \quad , \tag{6.3}$$

the last of which shows that anti-plane strain is locally volume-preserving.

For anti-plane strain the field equations of Section 2 can be reduced for a compressible material to three differential equations for the single unknown u , and in the incompressible case to three differential equations for the two unknowns u and p. It is thus not surprising that not all materials

— not even all homogeneous, isotropic ones — can sustain general states of anti-plane strain in the absence of body forces.[1] One class of incompressible materials which do have this property consists of those whose strain energy densities depend only on I_1:

$$W = W\left(I_1\right), \quad W(3) = 0 \ . \tag{6.4}$$[2]

Such materials are isotropic and include the neo-Hookean material, in which W is linear in I_1, as a special case. If (6.4) holds, the governing equations (2.6), (2.19), (6.2), (6.3) and the third of (2.3) can be consistently reduced to the following single differential equation[3] for the out-of-plane displacement $u\left(x_1, x_2\right)$:

$$\frac{\partial}{\partial x_\alpha}\left[W'\left(3 + |\nabla u|^2\right)\frac{\partial u}{\partial x_\alpha}\right] = 0 \qquad \text{on } \mathcal{R}_o \ , \tag{6.5}$$

where, as in the preceding section, $\mathcal{R}_o$ denotes an open cross-section of the undeformed cylinder, and the prime indicates differentiation with respect to the argument of W.

---

[1] A necessary and sufficient condition that a homogeneous, isotropic, incompressible elastic material admit general states of anti-plane strain is derived in [18]; the corresponding condition for compressible materials is considerably more restrictive and may be found in [19].

[2] The requirement that W should vanish in the undeformed state is a convenient normalization.

[3] Details of this reduction are included in [18]; it is assumed that the tractions on the lateral surface of the cylinder are independent of x_3. See also [6].

The components of nominal stress $\underset{\sim}{\sigma}$ and true stress $\underset{\sim}{\tau}$ are given by

$$\tau_{3\alpha} = \tau_{\alpha 3} = \sigma_{3\alpha} = \sigma_{\alpha 3} = 2W'\left(3 + |\underset{\sim}{\nabla} u|^2\right) u_{,\alpha} ,$$

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta} = \sigma_{33} = 0 , \qquad (6.6)^1$$

$$\tau_{33} = 2W'\left(3 + |\underset{\sim}{\nabla} u|^2\right) |\underset{\sim}{\nabla} u|^2 .$$

If $u = kx_2$, where k is constant, then (6.5) is satisfied and the deformation is one of simple shear.[2] The relation between the shear stress $\tau_{23} \equiv \tau$ and the amount of shear k is found from (6.6) to be

$$\tau = 2W'\left(3 + k^2\right) k \qquad (6.7)$$

in simple shear; $\mu = 2W'(3)$ is the shear modulus for infinitesimal deformations. The shear modulus at arbitrary shear k is $2W'\left(3 + k^2\right)$ and is assumed to be positive for all k.

The differential equation (6.5) is found to be elliptic at a solution u and at a point $\left(x_1, x_2\right)$ if

$$\left\{\frac{d}{dk}\left[W'\left(3 + k^2\right)k\right]\right\}_{k = |\underset{\sim}{\nabla} u(x_1, x_2)|} > 0 . \qquad (6.8)$$

From (6.7), this is equivalent to the condition that the curve of shear stress

---

[1] The hydrostatic pressure p occurring in (2.19) has been eliminated in terms of $|\underset{\sim}{\nabla} u|$ in the course of reducing the field equations to (6.5). See [18] or [6].

[2] Simple shear can, of course, be sustained without body forces in any elastic material.

versus amount of shear shall have positive slope[1] at $k = |\underset{\sim}{\nabla} u(x_1, x_2)|$.

For anti-plane strain, the components of the energy-momentum tensor are found from (4.11), (6.2) and (6.6) to be

$$
\begin{aligned}
P_{\alpha\beta} &= W\delta_{\alpha\beta} - 2W' u,_\alpha u,_\beta \ , \\
P_{\alpha 3} &= P_{3\alpha} = -2W' u,_\alpha \ , \quad P_{33} = W \ .
\end{aligned}
\qquad (6.9)^2
$$

When taken over closed cylindrical surfaces Γ of unit length which are coaxial with $\mathcal{R}$, the integrals J_i of (4.13) reduce to

$$J_\alpha = \oint_{\Gamma_o} \left(W\delta_{\alpha\beta} - 2W' u,_\alpha u,_\beta\right) N_\beta ds \ , \qquad (6.10)$$

$$J_3 = -\oint_{\Gamma_o} 2W' u,_\beta N_\beta ds \ , \qquad (6.11)$$

where Γ_o is the closed curve forming the boundary of the cross-section of Γ, and s is arc-length on Γ_o. If Γ_o encloses only points of $\mathcal{R}_o$ where u is twice continuously differentiable, one has the conservation laws

$$J_\alpha = 0 \ , \quad J_3 = 0 \ , \qquad (6.12)$$

as can be verified directly with the aid of (6.5) and the divergence theorem.

The function H of (4.17) reduces because of (6.9) to

$$H = -W\left(3 + |\underset{\sim}{\nabla} u|^2\right) + 2W'\left(3 + |\underset{\sim}{\nabla} u|^2\right)\left(\frac{\partial u}{\partial N}\right)^2 \ , \qquad (6.13)$$

---

[1] For the neo-Hookean material, $W'\left(3 + k^2\right) = \mu$ = constant, (6.5) is Laplace's equation, and (6.8) holds for all k.

[2] W and W' are to be evaluated at $3 + |\underset{\sim}{\nabla} u|^2$.

where $\partial u/\partial N = u,_{\alpha} N_{\alpha}$.

A special subclass of hypothetical incompressible isotropic materials[1] which satisfy (6.4) are those for which

$$W(I_1) = \frac{\mu}{2b}\left\{\left[1 + \frac{b}{n}(I_1 - 3)\right]^n - 1\right\} \ , \quad I_1 \geq 3 \ , \qquad (6.14)^2$$

where $\mu > 0$, $b > 0$ and $n > 0$ are material constants. The τ - k relation for this class of materials is

$$\tau = \mu\left(1 + \frac{b}{n}k^2\right)^{n-1} k \ . \qquad (6.15)$$

Graphs of τ vs k for various values of n are shown in Fig. 2. The ellipticity condition (6.8) is always satisfied if $b>0$, $n \geq 1/2$. If $b>0, 0<n<1/2$, ellipticity of (6.5) fails if

$$|\underset{\sim}{\nabla} u(x_1, x_2)| \geq \sqrt{\frac{n}{(1 - 2n)b}} \ . \qquad (6.16)$$

It will be assumed henceforth that $0<n<1/2$. For the special materials characterized by (6.14), formula (6.13) for H reduces to

$$H = \frac{\mu}{2b} + \mu\left(1 + \frac{b}{n}|\underset{\sim}{\nabla} u|^2\right)^{n-1}\left[\left(\frac{\partial u}{\partial N}\right)^2 - \frac{1}{2n}|\underset{\sim}{\nabla} u|^2 - \frac{1}{2b}\right] \ . \qquad (6.17)$$

For weak solutions in anti-plane strain which involve equilibrium shocks, both the material and the spatial shock surfaces are cylindrical, and

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[1] These materials have been considered in connection with a crack problem in [6].

[2] One always has $I_1 \geq 3$ in locally volume-preserving deformations.

a material shock intersects the cross-section $\mathcal{R}_o$ in a curve C. The traction jump condition (3. 3) can be seen from (6. 6) to reduce to

$$\left[W'\left(3+|\underset{\sim}{\nabla}u|^2\right)u,_\alpha\right]_-^+ N_\alpha = 0 \qquad \text{on } C , \tag{6.18}$$

while the conditions (3. 5) for continuity of the tangential displacement gradients become

$$[u,_\alpha]_-^+ L_\alpha = 0 \qquad \text{on } C , \tag{6.19}$$

where $\underset{\sim}{N}$ and $\underset{\sim}{L}$ are unit vectors normal and tangent to C, with $\underset{\sim}{N}$ pointing into the positive side of C.

From (6. 19), one concludes that there is a scalar g such that

$$\bar{u},_\alpha = \overset{+}{u},_\alpha + gN_\alpha \qquad \text{on } C . \tag{6.20}$$[1]

Let ϵ be introduced by

$$1-\epsilon = \frac{W'\left(3+|\underset{\sim}{\nabla}\overset{+}{u}|^2\right)}{W'\left(3+|\underset{\sim}{\nabla}\bar{u}|^2\right)} ; \tag{6.21}$$

thus ϵ is the relative change in the shear modulus across the shock. Since it has been assumed that $W'>0$, one has $\epsilon<1$. The traction continuity condition (6. 18) can be written as

$$(1-\epsilon)\overset{+}{u},_\alpha N_\alpha = \bar{u},_\alpha N_\alpha \qquad \text{on } C \tag{6.22}$$

whence, from (6. 20)

---

[1]Equation (6. 20) is the analog in anti-plane strain of (5. 10) for plane strain.

$$g = -\epsilon \overset{+}{u}_{,\alpha} N_\alpha = -\epsilon \frac{\partial \overset{+}{u}}{\partial N} . \tag{6.23}$$

From (6.20), (6.23) one has

$$|\nabla \bar{u}|^2 = |\nabla \overset{+}{u}|^2 - \left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2 (2 - \epsilon)\epsilon \qquad \text{on } C , \tag{6.24}$$

so that ($\bar{6}$.21) reduces to

$$W'(3 + \overset{+}{k}{}^2) = (1 - \epsilon) W'\left(3 + \overset{+}{k}{}^2 - \left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2 (2 - \epsilon)\epsilon\right), \qquad \text{on } C \tag{6.25}$$

where the abbreviations

$$\overset{\pm}{k} = |\nabla \overset{\pm}{u}|^2 \tag{6.26}$$

have been introduced. If $\nabla \overset{+}{u}$, and therefore $\overset{+}{k}$, is regarded as given, (6.25) furnishes one equation for the determination of $\partial \overset{+}{u}/\partial N = \nabla \overset{+}{u} \cdot N$ as a function of ϵ. According to (6.20) and (6.23), one would thus expect to find a one-parameter family of gradients $\nabla \bar{u}$ which can be "connected" to $\nabla \overset{+}{u}$ via the shock conditions;[1] ϵ — the shock strength — is to be chosen as the indexing parameter.[2]

For weak shocks in anti-plane strain, $|\epsilon| \ll 1$, and an analysis parallel to, but much simpler than, that carried out in the preceding section shows that the limiting value as $\epsilon \to 0$ of $\partial \overset{+}{u}/\partial N$ satisfies

---

[1] It is possible to show that the local existence of a solution $\partial \overset{+}{u}/\partial N$ of (6.25) for some $\epsilon \neq 0$ requires the material to be such that the ellipticity condition (6.8) should fail for some value of the amount of shear k.

[2] Note that the physical meaning of ϵ in anti-plane strain is not the same as that of its counterpart for plane deformations; compare (6.21), (5.8).

$$\left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2_{\epsilon = 0} = \frac{-W'(3 + \overset{+}{k}{}^2)}{2W''(3 + \overset{+}{k}{}^2)} . \qquad (6.27)$$

This result, in turn, can be used to show that the limiting direction of the shock as $\epsilon \to 0$ is that of a characteristic curve of the partial differential equation (6.5). There are in general two characteristic directions associated with a point (x_1, x_2) at which $\underset{\sim}{\nabla} u(x_1, x_2)$ is such that ellipticity has failed, and these characteristic directions are symmetric with respect to $\underset{\sim}{\nabla} u(x_1, x_2)$.

By carrying out the appropriate small - $|\epsilon|$ approximations using (6.13) and (6.25), one obtains the following weak-shock formula for $[H]^+_-$:

$$[H]^+_- = \left\{\frac{2}{3}\left(\frac{\partial \overset{+}{u}}{\partial N}\right)^4_{\epsilon = 0} W''' + \left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2_{\epsilon = 0} W''\right\} \epsilon^3 + O(\epsilon^4) \text{ as } \epsilon \to 0 , \qquad (6.28)$$

where W'' and W''' are to be evaluated at $3 + \overset{+}{k}{}^2$, and $\overset{+}{k}$ is given in (6.26). In this form, the approximation (6.28) is analogous to the plane strain result (5.36), (5.38). In the present circumstances, (6.27) can be used to further reduce (6.28) to

$$[H]^+_- = \frac{1}{3}\left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2_{\epsilon = 0} \left\{3(W'')^2 - W'W'''\right\} \epsilon^3 + O(\epsilon^4) \text{ as } \epsilon \to 0 , \qquad (6.29)$$

where, again, the derivatives of W are evaluated at $3 + |\nabla \overset{+}{u}|^2$.

Now consider a quasi-static time-dependent family of equilibrium shocks in anti-plane strain, and let the shock strength at time t associated with this family be $\epsilon(t)$. If the shock is sufficiently weak at time t, and if the dissipation inequality (4.19) holds, then (6.29) shows that

$$\left\{3[W''(I_1)]^2 - W'(I_1)W'''(I_1)\right\}\epsilon^3(t) \leq 0 \quad , \tag{6.30}$$

where $I_1 = 3 + \overset{+}{k}{}^2(t)$.

For the special material characterized by the strain energy density (6.14), one verifies easily that

$$3(W'')^2 - W'W''' = -\frac{\mu^2 b^2(1-2n)}{4n^2}\left[1 + \frac{b}{n}(I_1 - 3)\right]^{2n-4} \quad . \tag{6.31}$$

Since $b > 0$, $0 < n < 1/2$, this quantity is negative, and (6.30) thus requires that $\epsilon(t)$, which is always less than unity, shall also be nonnegative for this material, at least for sufficiently weak shocks. Reference to (6.21) then shows that the shear modulus $2W'(3 + k^2)$ is smaller on the positive — or "upstream" — side of the shock[1] than on the downstream side. Moreover, (6.24) shows that, for a dissipative weak shock in the particular material under consideration, the downstream side carries the smaller value of $|\underset{\sim}{\nabla} u|^2$, since $0 < \epsilon < 1$.

For the special strain energy density (6.14), the basic equation (6.25) governing the shock transition can in fact be solved explicitly and the conclusions above can be validated for all equilibrium shocks — not merely weak ones. The result of solving (6.25) in this case is

$$\left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2 = \frac{n}{b}\left(1 + \frac{b}{n}\overset{+}{k}{}^2\right)\frac{1 - (1-\epsilon)^{\frac{1}{1-n}}}{\epsilon(2-\epsilon)} \quad , \quad \epsilon < 1 \ . \tag{6.32}$$

---

[1] Recall that the velocity $\underset{\sim}{V}$ of the quasi-statically moving shock C points into the "positive" side of C.

Since

$$\left(\frac{\partial \overset{+}{u}}{\partial N}\right)^2 = (\nabla \overset{+}{u} \cdot N)^2 \leq |\nabla \overset{+}{u}|^2 = \overset{+}{k}{}^2 \quad , \tag{6.33}$$

one must have

$$\frac{n}{b}\left(1 + \frac{b}{n}\overset{+}{k}{}^2\right)\frac{1 - (1-\epsilon)^{\frac{1}{1-n}}}{\epsilon(2-\epsilon)} \leq \overset{+}{k}{}^2 \quad , \tag{6.34}$$

if a shock of strength ϵ is to exist in the presence of a displacement gradient of magnitude $\overset{+}{k}$ on the positive side of the shock. Thus, for a given value of the material constants b and n, only those points in the ϵ, $\overset{+}{k}$ – plane which lie above the curve in Fig. 3 – the admissible region – correspond to equilibrium shocks in anti-plane strain.

Finally, one can compute $[H]_-^+$ exactly for the special material at hand from (6.13), (6.14) and (6.32). One finds

$$[H]_-^+ = \frac{\mu}{2b}\left(1 + \frac{b}{n}\overset{+}{k}{}^2\right)^n f(\epsilon) \quad , \tag{6.35}$$

where

$$f(\epsilon) = \frac{1}{2-\epsilon}\left\{2n - 2 + \epsilon + (1-\epsilon)^{\frac{n}{1-n}} + (1-2n)(1-\epsilon)^{\frac{1}{1-n}}\right\}. \tag{6.36}$$

It can then be shown that $f(\epsilon) < 0$ for $0 < \epsilon < 1$, $f(0) = 0$, and $f(\epsilon) > 0$ for $-\infty < \epsilon < 0$. The dissipation inequality (4.19) would then again require $0 < \epsilon < 1$ for all (not just weak) shocks. That portion of the admissible region of Fig. 3 which corresponds to dissipative shocks is indicated by shading. Note that ellipticity is always lost on the upstream side of the shock when the dissipation condition holds.

7. A thermodynamic basis for the dissipation inequality.

The present section is concerned with a thermodynamic argument[1] in support of the dissipation inequality (4. 19). For simplicity, only compressible materials will be considered.

Consider a material characterized by an internal energy $\epsilon = \epsilon(\underset{\sim}{F}, \eta)$ (measured per unit of mass) which depends only on the deformation gradient tensor $\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{x}, t)$ and the entropy per unit mass $\eta = \eta(\underset{\sim}{x}, t)$, and for which the nominal stress tensor $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{x}, t)$ and the temperature $\theta = \theta(\underset{\sim}{x}, t) > 0$ are given by[2]

$$\sigma_{ij} = \rho_o \frac{\partial \epsilon}{\partial F_{ij}}(\underset{\sim}{F}, \eta) \quad , \quad \theta = \frac{\partial \epsilon}{\partial \eta}(\underset{\sim}{F}, \eta) \quad . \tag{7. 1}$$

Here ρ_o is the (constant) mass per unit undeformed volume, and, as in the preceding sections, $\underset{\sim}{x}$ is the position vector to a particle in the undeformed state. The internal energy density ϵ is assumed to be infinitely differentiable with respect to $\underset{\sim}{F}$ and η, and also to be such that the second of (7. 1) is uniquely invertible to give

$$\eta = \hat{\eta}(\underset{\sim}{F}, \theta) \tag{7. 2}$$

as an infinitely differentiable function of $\underset{\sim}{F}$ and θ for every nonsingular $\underset{\sim}{F}$ and every $\theta > 0$. Let

$$\psi = \psi(\underset{\sim}{F}, \theta) = \epsilon(\underset{\sim}{F}, \hat{\eta}(\underset{\sim}{F}, \theta)) - \theta\hat{\eta}(\underset{\sim}{F}, \theta) \tag{7. 3}$$

---

[1] I am indebted to Professor James R. Rice for the substance of the argument given here.

[2] In the terminology of §80 of [20], such a material is called perfect.

be the <u>free energy per unit mass</u>. One shows easily from (7.1) - (7.3) that

$$\sigma_{ij} = \rho_o \frac{\partial \psi}{\partial F_{ij}} (\underset{\sim}{F}, \theta) \;, \quad \eta = - \frac{\partial \psi}{\partial \theta} (\underset{\sim}{F}, \theta) \;. \tag{7.4}$$

As in section 3, it is assumed that, for each time t in the interval of interest, the displacement field $\underset{\sim}{u}$ is continuous and piecewise continuously differentiable on the region $\mathcal{R}$ occupied by the undeformed body, so that $\underset{\sim}{F} = \underset{\sim}{\nabla} \underset{\sim}{u}$ is piecewise continuous on $\mathcal{R}$ for each t. It is also assumed that η is piecewise continuous on $\mathcal{R}$ for each t. Finally, $\underset{\sim}{u}, \underset{\sim}{F}$ and η are presumed to be continuously differentiable in t for each $\underset{\sim}{x}$ in $\mathcal{R}$. It follows from (7.1) that the smoothness properties of $\underset{\sim}{\sigma}$ and θ are the same as those of $\underset{\sim}{F}$ and η.

Let $\mathcal{D}$ be an arbitrary regular subregion of $\mathcal{R}$, and let $\mathcal{D}_t^*$ be the image at time t of $\mathcal{D}$ under the motion $\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}, t)$. The <u>balance of energy</u> is postulated as follows:

$$\frac{d}{dt} \left\{ \int_{\mathcal{D}} \rho_o \epsilon \, dV + \frac{1}{2} \int_{\mathcal{D}} \rho_o \underset{\sim}{v}^2 dV \right\} = \int_{\partial \mathcal{D}} \underset{\sim}{s} \cdot \underset{\sim}{v} \, dA + \int_{\partial \mathcal{D}} \underset{\sim}{h} \cdot \underset{\sim}{N} \, dA \;, \tag{7.5}$$

for every $\mathcal{D}$. Here $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x}, t)$ is the particle velocity, $\underset{\sim}{s} = \underset{\sim}{\sigma} \underset{\sim}{N}$ is the nominal traction acting on the boundary $\partial \mathcal{D}_t^*$ of $\mathcal{D}_t^*$, and $\underset{\sim}{h}$ is the heat flux (measured per unit area of $\partial \mathcal{D}$) acting on $\partial \mathcal{D}_t^*$. Thus the left side represents the rate of increase of the sum of internal and kinetic energy, while the right side is the sum of the rate of work of the tractions and the rate of heat flow into $\mathcal{D}_t^*$ across its boundary.[1]

---

[1] Body forces and internal heat sources are assumed to be absent.

The rate of entropy production $\dot{\Gamma}(t)$ associated with the motion is defined by

$$\dot{\Gamma} = \frac{d}{dt} \int_{\mathcal{D}} \rho_o \eta \, dV - \int_{\partial\mathcal{D}} \frac{1}{\theta} \underset{\sim}{h} \cdot \underset{\sim}{N} \, dA \ . \tag{7.6}$$

It is further postulated that

$$\dot{\Gamma}(t) \geq 0 \tag{7.7}$$

for every $\mathcal{D}$, for every motion and for each time t.

Suppose now that the temperature is uniform throughout the body and independent of time $\theta(\underset{\sim}{x}, t) = \theta_o$ = constant for every $\underset{\sim}{x}$ and $\mathcal{R}$ and all t. By eliminating the heat flux integral between (7.5) and (7.6) and making use of the definition (7.3), one finds that

$$\dot{\Gamma} = \frac{1}{\theta_o} \left\{ \int_{\partial\mathcal{D}} \underset{\sim}{s} \cdot \underset{\sim}{v} \, dA - \frac{d}{dt} \left[\int_{\mathcal{D}} \rho_o \psi(\underset{\sim}{F}, \theta_o) \, dV + \frac{1}{2} \int_{\mathcal{D}} \rho_o \underset{\sim}{v}^2 \, dV \right] \right\} . \tag{7.8}$$

Thus the entropy production rate is proportional to the excess of the rate of work of the forces external to $\mathcal{D}_t^*$ over the rate of increase of the sum of the free energy and the kinetic energy.

A comparison of the first of (7.4) with (2.9) shows that the material presently under consideration may be identified with the (compressible) elastic material of section 2 by writing

$$W(\underset{\sim}{F}) = \rho_o \psi(\underset{\sim}{F}, \theta_o) \ , \tag{7.9}$$

as long as the motion is isothermal. For isothermal quasi-static motions of the kind considered in section 4, one neglects the contribution of the

kinetic energy in (7.8) and thus obtains, with the aid of (7.9), the result

$$\dot{\Gamma} = \frac{1}{\theta_o}\left\{ \int_{\partial\mathcal{D}} \underset{\sim}{s} \cdot \underset{\sim}{v} dA - \dot{U} \right\}, \tag{7.10}$$

where

$$U(t) = \int_{\mathcal{D}} W(\underset{\sim}{F}(\underset{\sim}{x}, t)) dV \ . \tag{7.11}$$

Now suppose that the isothermal, quasi-static motion involves a shock with the properties described in section 4. If the material shock surface S_t intersects $\mathcal{D}$, the (4.16) applies and can be used to write (7.10) in the form

$$\dot{\Gamma} = -\frac{1}{\theta_o} \int_{\hat{S}_t} [H]_-^+ \, \underset{\sim}{V} \cdot \underset{\sim}{N} dA \ , \tag{7.12}$$

where $\hat{S}_t$ is the part of S_t lying in $\mathcal{D}$. The postulate (7.7) of nonnegative entropy production rate then leads immediately to (4.18), from which the dissipation inequality (4.19) follows.

If no shock is present, then $[H] = 0$ and (7.12) gives $\dot{\Gamma}(t) = 0$. Thus, although smooth quasi-static motions of an elastic body do not result in dissipation, weak solutions involving equilibrium shocks generally do.

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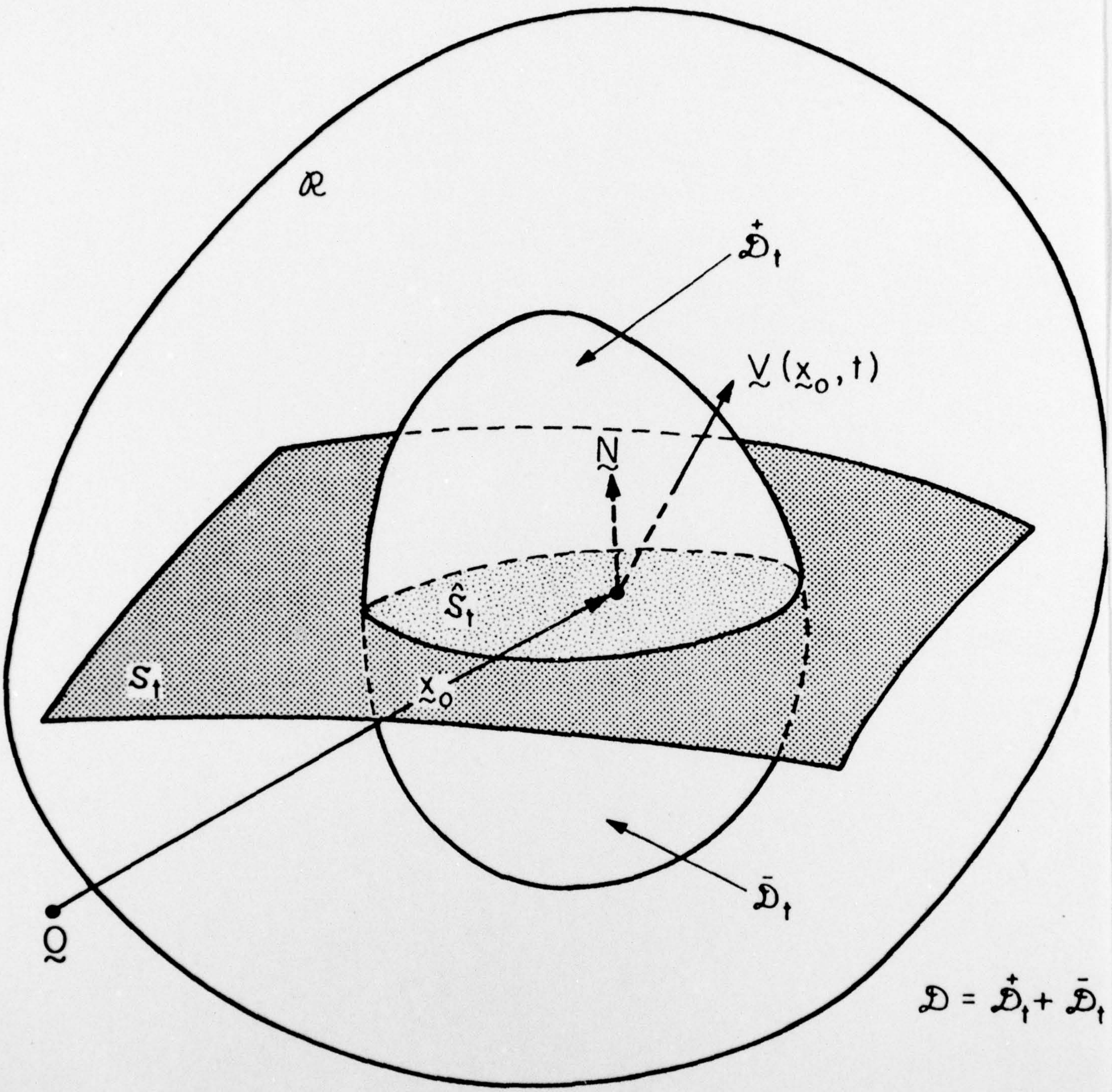

FIGURE 1. MOVING SHOCK S_t, WITH DOMAINS $\mathcal{D}$, $\overset{+}{\mathcal{D}}_t$, $\bar{\mathcal{D}}_t$

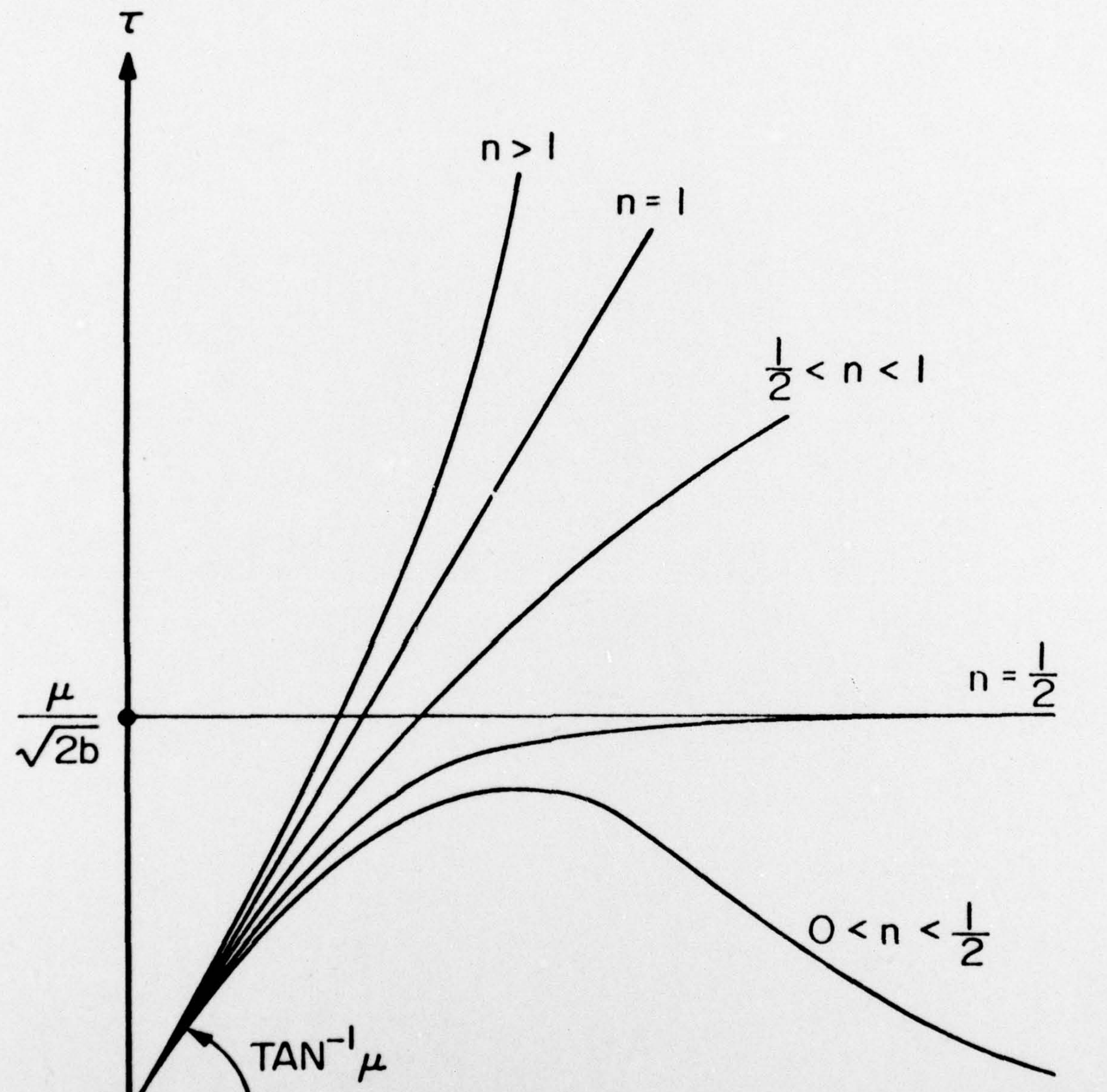

FIGURE 2. SHEAR STRESS τ VS. AMOUNT OF SHEAR k IN SIMPLE SHEAR (EQ. (6.15))

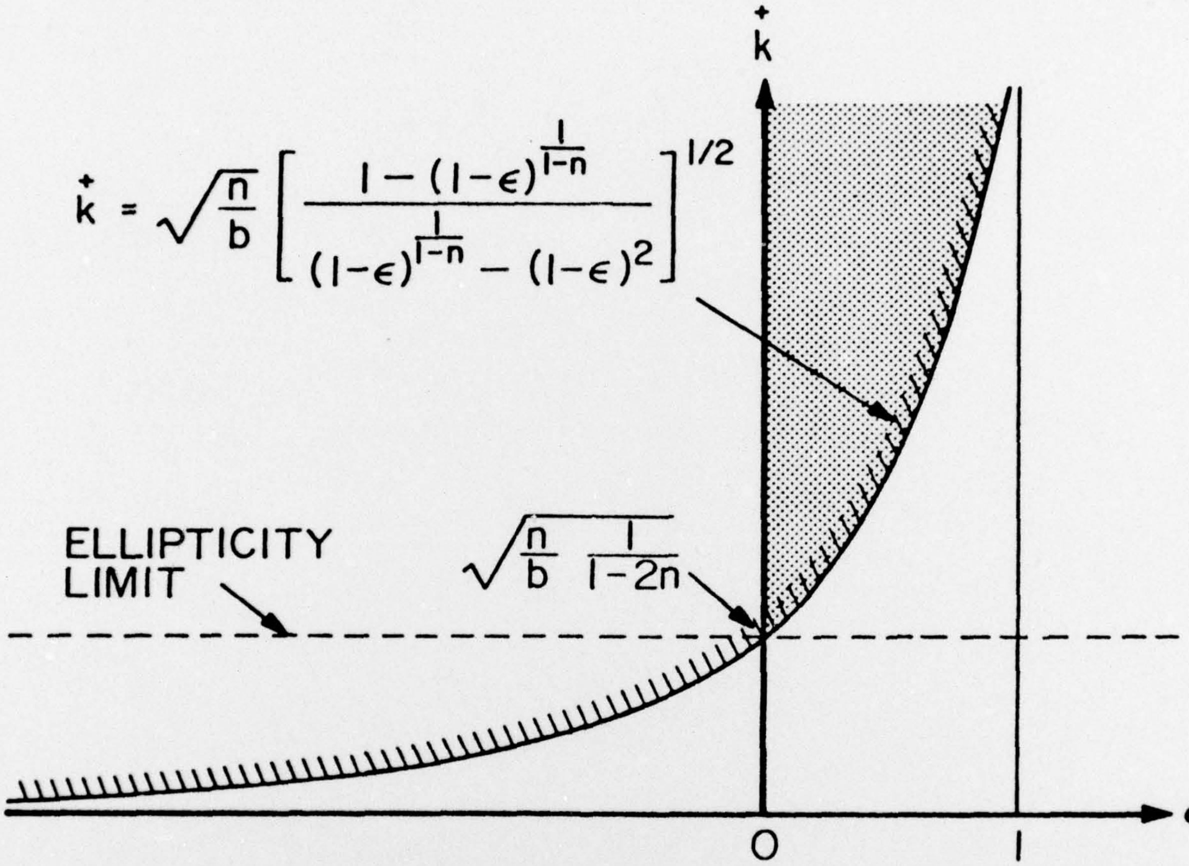

FIGURE 3. ADMISSIBLE REGION FOR EQUILIBRIUM SHOCKS IN ANTI-PLANE STRAIN FOR AN INCOMPRESSIBLE MATERIAL CHARACTERIZED BY (6.14) WITH $0<n<1/2$.

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13. ABSTRACT

Equilibrium fields with discontinuous displacement gradients can occur in finite elasticity for certain materials. The presence of such "equilibrium shocks" affects the energy balance in the elastostatic field, and the present paper is concerned with a notion of dissipation associated with this energy balance. A dissipation inequality is proposed for three-dimensional equilibrium shocks for both compressible and incompressible materials. The consequences of this inequality are studied for weak shocks in plane strain for compressible materials and for shocks of arbitrary strength in anti-plane strain for a class of incompressible materials. A thermodynamic argument for the dissipation inequality is also given.

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